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Temperature distributions in fins with uniform and non-uniform heat generation and non-uniform heat transfer coefficient

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Abstract—Temperature distributions in a straight fin of rectangular profile (or in a cylindrical fin) with uniform and non-uniform internal heat-generation characteristics and non-uniform heat transfer coefficients are derived analytically. The heat transfer coefficient is assumed to be a power function of the difference between the temperature of the fin and that of the fluid surrounding it. The power of this function is taken as being equal to -1, 0, 1 and 2. Non-uniform internal heat generation is assumed to depend on the fin temperature and this dependency is expressed in a polynomial equation up to the third degree. The results obtained for uniform internal heat generation are also presented in tabular and graphical forms and an interpolation method is proposed to determine the fin temperature and the fin effectiveness if the foregoing power is a fraction between -1 and 2.

INTRODUCTION

ANALYTICAL studies dealing with the determination of the temperature distribution in a fin with internal heat generation are rarely to be found in the literature [1–5]. This determination is of practical significance in the field of nuclear engineering [1] and of scientific measurements [6] (i.e. hot-wire anemometers and resistance temperature transducers).

Assuming that the internal heat generation and heat transfer coefficient are uniform, Minkler and Rouleau [1] analytically derived the temperature distribution in rectangular and triangular fins, and Liu [2] those in optimum rectangular and circular fins. Hung and Appl [3] presented approximate analytically calculated temperature distributions in the fins with temperature-dependent thermal properties and internal heat generation.

For most practical applications, the heat transfer coefficient is not uniform but a power function of the difference between the temperature of a heat transferring surface and that of the fluid surrounding this surface. It is expressed by

$$h = a\theta^n \tag{1}$$

in which a and n are constants. Typical values for n are -0.25, 0.25 and 2 for film-type condensation, natural convection and nucleate boiling, respectively.

The object of this work is the analytic derivations of the temperature distributions in a straight fin of rectangular profile or in a cylindrical fin with uniform and non-uniform internal heat-generation characteristics. To this end the heat transfer coefficient is taken as that given in equation (1) wherein $n = -1$, 0, 1 and 2.

Initially, internal heat generation in the fin is assumed to be constant and the temperature distributions in the fin and the fin effectiveness are determined as functions of dimensionless variables, and the results obtained are presented in graphical and tabular forms. An interpolation method is suggested for the approximate determination of the temperature distribution in the fin and the fin effectiveness if n is a fraction between -1 and 2. For these values of n, the temperature distribution in the fin cannot be analytically derived even for the simple condition in which no internal heat sources exist in the fin [7].

As a second step internal heat generation in the fin is assumed to be dependent on the temperature of the fin itself. This dependency is expressed in a polynomial equation up to third degree and the temperature distributions in the fin are derived herewith.

DIFFERENTIAL EQUATION OF TEMPERATURE DISTRIBUTION

A straight fin of rectangular profile or a cylindrical (i.e. pin) fin is now considered. For the analysis of such a fin, the following assumptions are made: one-dimensional steady-state heat conduction through the fin, constant thermal conductivity of the fin material, negligible heat transfer from the fin tip and a constant cross-sectional area for the fin. Internal heat generation in the fin is either constant or dependent on the temperature of the fin. The temperature of the fluid surrounding the fin is constant. The origin of the space coordinate x is at the fin base and positive x is toward the fin tip. The straight fin is infinitely long in the longitudinal direction. A unit length of this fin in this direction is being considered here. The heat transfer coefficient is given by equation (1).

For the assumptions made, the non-dimensional

NOMENCLATURE

A	cross-sectional area of a fin [m^2]
a	a given constant [W m^{-2} K$^{-(n+1)}$]
$a_1, \ldots, a_9$	constants defined in the text
Bi	modified Biot number at the base of a fin
b	modified fin parameter
C	integration constant
d, d_1	constants defined in the text
$e, \ldots, e_6$	constants defined in the text
$F(\mu/\alpha)$	Legendre's normal elliptic integral of the first kind
f	fin effectiveness
G	rate of heat flow from a fin [W]
$g, \ldots, g_3$	constants defined in the test [W m^{-3}, W m^{-3} K^{-1}, W m^{-3} K^{-2} and W m^{-3} K^{-3}, respectively]
h	heat transfer coefficient [W m^{-2} K^{-1}]
K	thermal conductivity of the fin material [W m^{-1} K^{-1}]
L	fin length [m]
M, m	constants defined in the text
N	generation number
n	a given constant
P	circumference of a cylindrical fin [m]
r	a constant defined in the text
Q	internal heat generation [W m^{-3}]
$q, \ldots, q_4$	constants defined in the text
$S, \ldots, S_3$	constants defined in the text [W m^{-3}, W m^{-3} K^{-1}, W m^{-3} K^{-2} and W m^{-3} K^{-3}, respectively]
$S_4, \ldots, S_7$	constants defined in the text
T	dimensionless temperature
t	temperature [K]
U	half fin thickness [m]
W	a constant defined in the text [m]
X	dimensionless space coordinate
x	space coordinate [m]
Y, Z	constants defined in the text.

Greek symbols

α	modular angle [rad]
β	a root of a polynomial equation
$\gamma, \ldots, \gamma_3$	constants defined in the text
θ	the difference between the temperature of a fin and that of the fluid surrounding it at point x [K]
μ	amplitude [rad].

Subscripts

b	refers to the fin base (i.e. $x = 0$)
e	refers to the fin tip (i.e. $x = L$)
f	refers to the fin
i	an index ($i = 1, 2, 3 \ldots$)
v	refers to the fluid.

differential equation of the temperature distribution in the fin then becomes

$$\frac{d^2T}{dX^2} - bT_b^{-n}T^{(n+1)} = -bNT_b. \tag{2}$$

The boundary conditions are expressed by

$$T = T_b = \theta_b/\theta_e \qquad \text{for } X = 0 \tag{3}$$

and

$$-\frac{\theta_e}{L}\frac{dT}{dX} = 0 \qquad \text{for } X = 1. \tag{4}$$

The non-dimensional parameters used in equations (2)–(4) are defined as

$$T = \theta/\theta_e \tag{5}$$

$$X = x/L \tag{6}$$

$$b = \frac{L^2h_b}{WK} = \frac{aL^2\theta_b^n}{WK} \tag{7}$$

$$N = \frac{QW}{h_b\theta_b} = \frac{QW}{a\theta_b^{(n+1)}} \tag{8}$$

in which

$$W = U \qquad \text{for the straight fin} \tag{9}$$

and

$$W = A/P \qquad \text{for the cylindrical fin.} \tag{10}$$

In accordance with the first boundary condition expressed in equation (3), the temperature difference at the fin base is equal to θ_b. The second boundary condition explicitly implies that no heat transfer takes place at the fin tip.

In order to solve equation (2), L, a, W, K, Q and n are assumed to be known. θ_b is the boundary value. θ_e is not an unknown value, but it can be determined only after the equation has been solved. By definition it is equal to θ_b/T_b. The foregoing implies that the values of b and N in equation (2) can be determined before solving it and the value of T_b after solving it.

The value of b given by equation (7) is the modified fin parameter. For the condition where $n = 0$ (i.e. a constant heat transfer coefficient), the square root of b (i.e. fin parameter or aspect number) is widely used in studies dealing with extended surfaces.

The value of N given by equation (8) is the generation number. It is the ratio of the total heat gen-

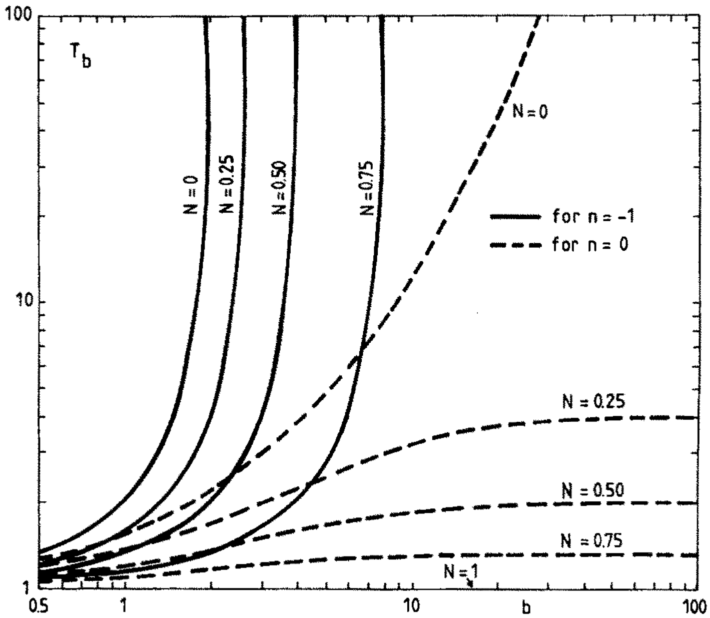

FIG. 1. T_b as a function of b and N for $n = -1$ and 0.

erated in the fin to the heat that would be dissipated from the fin if all of the fin was at the base temperature [1]. Accordingly, if $N = 1$ then all the heat generated in the fin is transferred to the fluid surrounding the fin, the temperature of the fin becomes uniform and no heat is conducted into the fin at its base since $dT/dX = 0$. Therefore, it follows from the foregoing that the generation number can only vary between 0 and 1. For $N = 0$, there is no internal heat generation in the fin, and for $N = 1$ internal heat generation in the fin is maximum.

SOLUTION OF THE DIFFERENTIAL EQUATION FOR UNIFORM HEAT GENERATION

In the case where $n = -1$

In this particular case, the right-hand side of equation (2) and the second term on the left-hand side are constants. The solution of this equation, which satisfies the boundary conditions expressed in equations (3) and (4), is then a straightforward matter and the dimensionless temperature distribution in the fin is given by

$$T = T_b\left(\frac{b}{2}(1-N)(X^2-2X)+1\right). \qquad (11)$$

From the above equation T_b (or θ_e) is calculated using the condition that $T = T_e = 1$ for $X = 1$

$$T_b = \frac{\theta_b}{\theta_e} = \left(1-\frac{b}{2}(1-N)\right)^{-1}. \qquad (12)$$

Thus it follows from equations (11) and (12) that T has only one value if the values of b, N and X are fixed. The dimensionless temperature at the fin base, T_b, is shown as a function of b and N in Fig. 1. In this figure, the asymptotic behaviour of T_b is obvious. In accordance with equation (12), an increment in $b(1-N)/2$ increases T_b. However, the maximum value of $b(1-N)/2$ should be less than 1. If $(1-b(1-N)/2)$ is zero, θ_e also becomes zero (i.e. $\theta_e = \theta_b(1-b(1-N)/2)$), and T (or T_b) is not defined. The foregoing implies that for a given value of N, the maximum value of b is determined by one of the two following inequalities derived with equation (12) and with the definition of the generation number:

$$b < \frac{2}{1-N} \qquad \text{for } 0 \leqslant N < 1 \qquad (13)$$

and

$$0 < b < \infty \qquad \text{for } N = 1. \qquad (14)$$

For $n = -1$, the heat flux on the surface of the fin all along its length is constant and is equal to a (see equation (1)). In accordance with equation (11), the temperature in the fin is constant when $N = 1$.

In the case where $n = 0$

In this particular case, the heat transfer coefficient is constant and equation (2) becomes a linear second-order differential equation with constant coefficients. The right-hand side of the equation is constant. The solution of the equation is taken from an appropriate textbook [8]. Thus the dimensionless temperature distribution in the fin, which satisfies the boundary conditions expressed in equations (3) and (4), is given by

$$T = T_b\left\{(1-N)\frac{\cosh\{\sqrt{b}(1-X)\}}{\cosh\sqrt{b}}+N\right\}. \qquad (15)$$

From equation (15) T_b (or θ_e) is determined using the condition that $T = 1$ for $X = 1$

$$T_b = \frac{\theta_b}{\theta_e} = \left(\frac{1-N}{\cosh\sqrt{b}} + N\right)^{-1}. \quad (16)$$

It follows from equations (15) and (16) that T has a single value if the values b, N and X are fixed. The dimensionless temperature at the fin base, T_b, is shown as a function of b and N in Fig. 1. In this figure the asymptotic behaviour of T_b is obvious if $0 < N < 1$. For a given value of N and for comparatively small values of b, an increment in b increases T_b. If b is increased beyond a certain value, T_b approaches $1/N$. If $N = 0$ the magnitude of T_b is not restricted, as can be deduced from equation (16). It follows from equation (15) that the temperature in the fin is constant for $N = 1$ (i.e. $T = T_b = 1$).

Contrary to the case analysed for $n = -1$, no restriction applies to the magnitude of b for $0 \leqslant N < 1$ if $n = 0$.

In the case where $n = 1$

In order to calculate the dimensionless temperature distribution in the fin for this case, equation (2) is integrated twice. The first integration of this equation can be carried out in accordance with the procedure explained in ref. [7]. Omitting the details, the result of this integration including the determination of the integration constant (using the boundary condition given in equation (4)) is presented below for all values of n excluding $n = -2$

$$\frac{dT}{dX} = -\left\{\frac{2b}{n+2}T_b^{-n}(T^{(n+2)} + \{n+2\}NT_b^{(n+1)}\{1-T\}-1)\right\}^{0.5}. \quad (17)$$

For $n = 1$ and after rearrangement, equation (17) reduces to

$$\frac{dT}{\{\gamma(T^3+3NT_b^2\{1-T\}-1)\}^{0.5}} = -dX \quad (18)$$

in which

$$\gamma = \frac{2b}{3T_b}. \quad (19)$$

The integration of equation (18) yields [9]

$$mF(\mu/\alpha) = -X + C \quad (20)$$

where $F(\mu/\alpha)$ is Legendre's normal elliptic integral of the first kind. Its value is determinable if the amplitude μ and the modular angle α in it are known. μ is a function of T and α is a constant. In ref. [10] $F(\mu/\alpha)$ is tabulated for $0 < \mu \leqslant \pi/2$, and is given as an infinite series for $0 \leqslant \mu \leqslant \pi$ in refs. [7, 9].

After the determination of the integration constant C in equation (20) (using the boundary condition expressed in equation (3)), the dimensionless temperature distribution in the fin becomes

$$mF(\mu/\alpha) = -X + mF(\mu_b/\alpha). \quad (21)$$

In order to utilize equation (21), only $\mu = \mu(T)$, m and α should be calculated since $\mu_b = \mu(T_b)$. For this purpose the roots of the cubic equation in equation (18) are required. These roots are expressed by

$$\beta_1 = 1 \quad (22)$$

and

$$\beta_{2,3} = -0.5 \pm (3NT_b^2 - 0.75)^{0.5}. \quad (23)$$

For the evaluations of β_2 and β_3, the value of T_b should be known. This value is determined using the equation given below, which is obtained from equation (21) using the condition that $\mu = \mu_e$ for $X = 1$

$$mF(\mu_b/\alpha) - mF(\mu_e/\alpha) = 1. \quad (24)$$

If all the three roots of the cubic equation are real and $\beta_1 > \beta_2 > \beta_3$, the formulae to calculate $\mu(T)$, m and α are presented below. These formulae are adapted from those given in ref. [9] and their derivations are therefore omitted here

$$\mu = \arcsin\left\{\left(\frac{\beta_1 - T}{\beta_2 - T}\right)^{0.5}\right\} \quad \text{for } 0 \leqslant \mu \leqslant \pi/2 \quad (25a)$$

$$m = 2\{\gamma(\beta_1 - \beta_3)\}^{-0.5} \quad (26a)$$

$$\alpha = \arcsin\left\{\left(\frac{\beta_2 - \beta_3}{\beta_1 - \beta_3}\right)^{0.5}\right\} \quad (27a)$$

μ_b and μ_e are obtained from equation (25a), noting that $\mu = \mu_b$ for $T = T_b$, $\mu = \mu_e$ for $T = T_e = 1$ and $\beta_1 = 1$

$$\mu_b = \arcsin\left\{\left(\frac{\beta_1 - T_b}{\beta_2 - T_b}\right)^{0.5}\right\} \quad (28a)$$

$$\mu_e = \arcsin 0 = 0. \quad (29a)$$

For $\mu_e = 0$, $F(\mu_e/\alpha)$ becomes zero; accordingly equation (24) reduces to

$$mF(\mu_b/\alpha) = 1. \quad (30a)$$

If the cubic equation has one real and two complex roots, the formulae to calculate $\mu(T)$, m and α are

$$\mu = \arccos\left(\frac{T - r - M\cot Z}{T - r + M\tan Z}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \quad (25b)$$

$$m = -M(\tan Z + \cot Z)\left\{\frac{4\gamma M^3}{\sin^3(2Z)}\right\}^{-0.5} \quad (26b)$$

$$\alpha = Z \quad (27b)$$

in which

$$\tan(2Z) = \frac{M}{\beta_1 - r} \quad \text{for } 0 < 2Z < \pi. \quad (31)$$

In order to derive M and r in μ, m amd α, the cubic equation in the denominator of equation (18) is expressed as

$$\gamma(T-1)(T^2+T+1-3NT_b^2) = \gamma(T-1)\{(T-r)^2+M^2\}. \quad (32)$$

The last term on the left-hand side of equation (32) should be identical to the last term on the right-hand side. This yields

$$r = -0.5 \quad (33)$$

$$M = (0.75-3NT_b^2)^{0.5}. \quad (34)$$

By definition, M should always be a positive number and r a real number. From equation (25b) μ_b is predicted using the condition that $\mu = \mu_b$ for $T = T_b$

$$\mu_b = \arccos\left(\frac{T_b-r-M\cot Z}{T_b-r+M\tan Z}\right). \quad (28b)$$

In order to calculate μ_e, first the values of r and T for $X = 1$ (i.e. $r = -0.5$ and $T_e = 1$) are introduced into equation (25b). Thereafter the resulting numerical value (i.e. 1.5) both in the numerator and the denominator of the fraction in this equation is replaced by $M/\tan(2Z)$ as calculated with equation (31) (i.e. $\beta_1 = 1$ and $r = -0.5$). If $\tan(2Z)$ and $\cot Z$ in the present equation are expressed as a function of $\tan Z$, the equation produces the value of μ_e after rearrangement

$$\mu_e = \arccos -1 = \pi. \quad (29b)$$

For $\mu_e = \pi$, equation (24) reduces to

$$mF(\mu_b/\alpha)-mF(\pi/\alpha) = 1. \quad (30b)$$

Relative to equations (25)–(30), the a-versions of the equations are valid if the cubic equation has three real roots and $1 > \beta_2 > \beta_3$. The b-versions of the equations are valid if the cubic equation has one real and two complex roots. The c-versions of the equations will apply in the case where $n = 2$.

If the cubic equation has three real roots and if one of these roots is greater than 1, the solution of equation (2) is again expressed in equation (21); this solution however appears to be trivial [9]. Therefore, the formulae to predict $\mu(T)$, m and α are not given here. The foregoing will be further clarified whilst presenting T_b as a function of b and N.

In order to calculate the dimensionless temperature in the fin, only the value of T_b is required. For the determination of T_b the following procedure is adopted. A value for T_b is assumed so M or β_2 and β_3 are then known. The value of m, α and μ_b is predicted with equations (26)–(28), respectively. Using the values of α and μ_b, $F(\mu_b/\alpha)$ is determined from the tabulated values of $F(\mu/\alpha)$ (or from the analytic expression of $F(\mu/\alpha)$). Also $F(\mu_b/\alpha)$ is determined from equation (30). The value of T_b is iterated until the calculated two $F(\mu_b/\alpha)$ values are identical.

Having found T_b, the evaluation of T for a given value of X is carried out with equation (21). To this end, first $F(\mu/\alpha)$ is determined with this equation since the values of m, α, X and $F(\mu_b/\alpha)$ are known. Thereafter the value of μ, which satisfies this $F(\mu/\alpha)$, is obtained from the tabulated values of $F(\mu/\alpha)$ or (from the analytic expression of $F(\mu/\alpha)$). T is then predicted with equation (25).

For the evaluation of X for a given T (i.e. $T_b \geq T \geq 1$), μ is first solved from equation (25), thereafter $F(\mu/\alpha)$ is solved from the tabulated values of $F(\mu/\alpha)$ (or from the analytic expression of $F(\mu/\alpha)$), and finally X is solved from equation (21). The foregoing method seems simpler than the previous one.

If the values of b and N are fixed, the value of T_b is determined. Since $T_b \geq T \geq 1$, T has only a single value if the values of b, N and X are fixed. The values of T for $X = 0$, 0.25, 0.50 and 0.75 were calculated as a function of b and N. The results obtained are tabulated in Table 1. For $X = 1$, $T = 1$. The values of N were taken equal to 0, 0.25, 0.50 and 0.75. For $N = 1$, $T = T_b$. The calculations were carried out on a programmable desk calculator with 224 program steps. In order to predict $F(\mu/\alpha)$, the analytic expression of $F(\mu/\alpha)$ was used. This expression is an infinite series. For large values of α and μ, the Gaußsche transformation [9] was used to reduce the values of α and μ; consequently the number of terms needed to calculate the foregoing series was very small. If α and μ are greater than $89\pi/180$ rad and if $F(\mu/\alpha)$ is evaluated from the tabulated values of $F(\mu/\alpha)$, the Gaußsche transformation is also needed since $F(\pi/2/\pi/2)$ is infinite.

The dimensionless temperature at the fin base, T_b, is shown as a function of b and N in Fig. 2. In this figure, the asymptotic behaviour of T_b is obvious if $1 > N > 0$. For a given value of N and for comparatively small values of b, an increment in b increases T_b. If b is increased beyond a certain value, T_b approaches asymptotically to $1/\sqrt{N}$ (see also Table 1). For $1/\sqrt{N} > T_b \geq 0.5/\sqrt{N}$, β_2 and β_3 are real roots and $1 > \beta_2 > \beta_3$ (see equations (22) and (23)). If $T_b = 1/\sqrt{N}$, $\beta_2 = 1$, and $F(\mu/\alpha)$ and $F(\mu_b/\alpha)$ become infinite (i.e. $\mu = \mu_b = \alpha = \pi/2$). For $\beta_1 = \beta_2 = 1$ the dimensionless temperature which is obtained from equation (25a) reduces to

$$T = \frac{\beta_1-\beta_2\sin^2\mu}{1-\sin^2\mu} = 1. \quad (35)$$

Thus the temperature in the fin is constant and consequently no heat is conducted into the fin at its base. Accordingly T_b is always less than $1/\sqrt{N}$ and β_2 less than 1. For $N = 0$, μ_b given by equation (28b) becomes

$$\mu_b = \arccos\left(\frac{T_b-2.732}{T_b+0.732}\right). \quad (36)$$

In accordance with equation (36), no restriction applies to the magnitude of T_b for $N = 0$ since μ_b varies between π and 0 if $1 < T_b < \infty$ and α is equal to $\pi/12$.

For $N = 1$ and $T_b = 1$, β_2 is equal to 1 and the

Table 1. $f\sqrt{Bi}$ and dimensionless temperature in the fin as a function of dimensionless variables for $n = 1$

	$N = 0$					$N = 0.25$					$N = 0.50$					$N = 0.75$				
		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$
b	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T
0.10	0.297	1.049	1.027	1.012	1.003	0.223	1.036	1.020	1.009	1.002	0.148	1.024	1.013	1.006	1.001	0.074	1.012	1.007	1.003	1.001
0.25	0.433	1.116	1.064	1.028	1.007	0.324	1.085	1.047	1.021	1.005	0.216	1.055	1.030	1.013	1.003	0.108	1.027	1.015	1.006	1.002
0.50	0.547	1.220	1.120	1.052	1.013	0.409	1.155	1.084	1.037	1.009	0.271	1.098	1.053	1.023	1.006	0.135	1.046	1.025	1.011	1.003
1	0.653	1.404	1.215	1.092	1.022	0.484	1.269	1.143	1.061	1.015	0.320	1.162	1.085	1.036	1.009	0.158	1.074	1.039	1.016	1.004
2	0.732	1.720	1.367	1.153	1.037	0.538	1.434	1.219	1.090	1.022	0.352	1.242	1.121	1.049	1.012	0.173	1.104	1.052	1.021	1.005
3	0.764	1.998	1.491	1.200	1.048	0.557	1.547	1.264	1.105	1.025	0.362	1.290	1.137	1.054	1.013	0.177	1.121	1.056	1.021	1.005
4	0.780	2.253	1.598	1.239	1.056	0.565	1.632	1.293	1.114	1.026	0.367	1.321	1.145	1.055	1.013	0.179	1.130	1.057	1.021	1.005
5	0.790	2.493	1.693	1.273	1.064	0.570	1.696	1.310	1.117	1.027	0.369	1.342	1.147	1.054	1.012	0.180	1.137	1.057	1.020	1.004
6	0.796	2.723	1.780	1.303	1.071	0.573	1.746	1.320	1.118	1.027	0.370	1.357	1.147	1.052	1.011	0.180	1.141	1.056	1.019	1.004
7	0.800	2.944	1.860	1.331	1.077	0.574	1.786	1.326	1.117	1.026	0.371	1.369	1.145	1.050	1.011	0.180	1.144	1.054	1.018	1.004
8	0.803	3.156	1.934	1.355	1.082	0.575	1.818	1.327	1.115	1.025	0.371	1.377	1.142	1.047	1.010	0.181	1.146	1.053	1.017	1.003
9	0.806	3.364	2.003	1.377	1.086	0.576	1.844	1.326	1.112	1.024	0.371	1.384	1.139	1.045	1.009	0.181	1.148	1.051	1.016	1.003
10	0.807	3.566	2.068	1.398	1.091	0.576	1.866	1.324	1.109	1.023	0.371	1.389	1.135	1.042	1.008	0.181	1.149	1.049	1.014	1.003
20	0.814	5.420	2.576	1.546	1.120	0.577	1.962	1.271	1.073	1.014	0.372	1.409	1.100	1.024	1.004	0.181	1.154	1.034	1.007	1.001
50	0.816	10.25	3.470	1.765	1.161	0.577	1.997	1.149	1.024	1.003	0.372	1.414	1.048	1.006	1.001	0.181	1.155	1.015	1.001	1.000
100	0.816	17.52	4.299	1.934	1.190	0.577	1.999	1.071	1.006	1.000	0.372	1.414	1.020	1.001	1.000	0.181	1.155	1.006	1.000	1.000

dimensionless temperature in the fin is again expressed in equation (35), which states the fact that the temperature in the fin is constant.

In the case where $n = 2$

For this case, the first integration of equation (2) is given by equation (17). After rearrangement and when $n = 2$, this equation reduces to

$$\frac{\mathrm{d}T}{\{\gamma_1(T^4+4NT_b^3\{1-T\}-1)\}^{0.5}} = -\mathrm{d}X \qquad (37)$$

in which

$$\gamma_1 = \frac{b}{2T_b^2}. \qquad (38)$$

The integration of equation (37) and the relationship required to determine T_b is again given by equations (21) and (24), respectively. In order to utilize these equations, $\mu(T)$, m and α in equation (21) should be known. For the determinations of $\mu(T)$, m and α, the roots of the polynomial equation in equation (37) are needed. β_1, one of the roots of this polynomial equation is always equal to 1, therefore the polynomial equation is reduced to

$$\gamma_1(T-1)(T^3+T^2+T+1-4NT_b^3) = 0. \qquad (39)$$

The roots of the cubic equation in equation (39) can be calculated with the formulae given in ref. [11]. Therefore, in accordance with these formulae, this cubic equation has one real and two complex roots (since d expressed in equation (41) always yields a positive real number). The real root is given by

$$\beta_2 = -\frac{2}{9d}+d-\frac{1}{3} \qquad (40)$$

in which

$$d = \left\{-\frac{1}{2}\left(\frac{20}{27}-4NT_b^3\right) + \frac{1}{2}\left(\left\{\frac{20}{27}-4NT_b^3\right\}^2+\frac{32}{729}\right)^{0.5}\right\}^{1/3}. \qquad (41)$$

Having found β_1 and β_2, $\mu(T)$, m and α can be derived from the formulae given in ref. [9]

$$\mu = \arccos\left(\frac{a_8-a_6T}{a_5T-a_7}\right) \quad \text{for } 0 \leqslant \mu \leqslant \pi \qquad (25c)$$

$$m = \frac{2a_4a_9}{\{-2\gamma_1a_2a_4(\beta_1-\beta_2)\}^{0.5}} \qquad (26c)$$

$$\alpha = \arcsin a_9 \qquad (27c)$$

in which

$$a_9 = (1+a_4^2)^{-0.5} \qquad (42)$$

$$a_8 = \frac{1}{2}(\beta_1+\beta_2)a_6-\frac{1}{2}(\beta_1-\beta_2)a_5 \qquad (43)$$

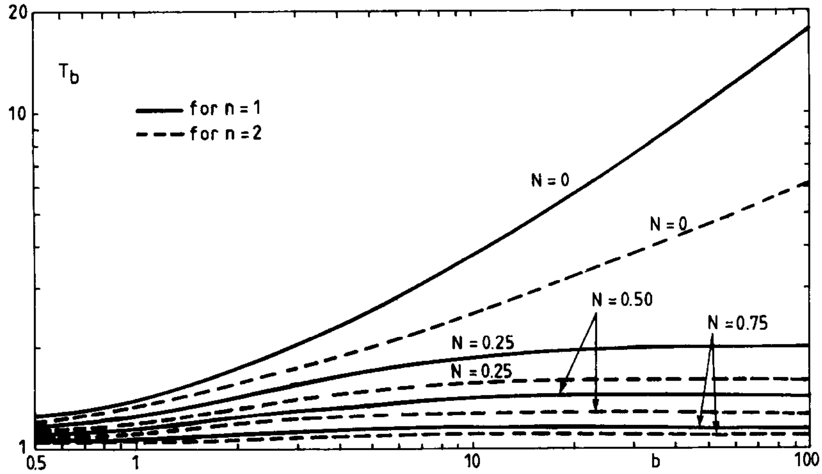

FIG. 2. T_b as a function of b and N for $n = 1$ and 2.

$$a_7 = \frac{1}{2}(\beta_1+\beta_2)a_5 - \frac{1}{2}(\beta_1-\beta_2)a_6 \tag{44}$$

$$a_6 = \beta_1 - a_1 + a_2 a_4 \tag{45}$$

$$a_5 = \beta_1 - a_1 - a_2/a_4 \tag{46}$$

$$a_4 = a_3 \pm (a_3^2+1)^{0.5} \tag{47}$$

$$a_3 = \frac{a_2^2+(\beta_1-a_1)(\beta_2-a_1)}{a_2(\beta_1-\beta_2)}. \tag{48}$$

In order to derive a_1 and a_2 in a_3–a_9, the fourth-order polynomial equation in equation (37) is expressed as

$$\gamma_1(T-1)(T-\beta_2)\{T^2+(\beta_2+1)T+\beta_2^2+\beta_2+1\}$$
$$= \gamma_1(T-1)(T-\beta_2)\{(T-a_1)^2+a_2^2\}. \tag{49}$$

The last term on the left-hand side of equation (49) should be identical to the last term on the right-hand side. This yields

$$a_1 = -(\beta_2+1)/2 \tag{50}$$

$$a_2 = (3\beta_2^2+2\beta_2+3)^{0.5}/2. \tag{51}$$

By definition, $a_2 > 0$ and $\gamma_1 a_4 < 0$. As will be explained later μ, m, α and a_1–a_9 are valid if $\beta_2 < 1$.

From equation (25c) μ_b is evaluated noting that $\mu = \mu_b$ for $T = T_b$

$$\mu_b = \arccos\left(\frac{a_8 - a_6 T_b}{a_5 T_b - a_7}\right). \tag{28c}$$

From equation (25c) μ_e is determined using the condition that $\mu = \mu_e$ for $T = T_e = 1$; introducing this value of T_e, the value of β_1 and a_5–a_8 given by equations (46)–(43) into equation (25c) and after the rearrangement, the latter equation yields

$$\mu_e = \arccos\ -1 = \pi. \tag{29c}$$

For $\mu_e = \pi$, equation (24) reduces to

$$mF(\mu_b/\alpha) - mF(\pi/\alpha) = 1. \tag{30c}$$

The determinations of T_b, T for a given X and X for a given T are carried out with the procedures adopted in the case where $n = 1$. As noted earlier, the c-versions of equations (25)–(30) apply in the case where $n = 2$.

If the values of b and N are fixed, the value of T_b is determined. Since $T_b \geqslant T \geqslant 1$, T has one single value if the values of b, N and X are fixed. Values of T for $X = 0$, 0.25, 0.50 and 0.75 are tabulated as a function of b and N in Table 2. Here N was taken equal to 0, 0.25, 0.5 and 0.75.

The dimensionless temperature at the fin base, T_b, is shown as a function of b and N in Fig. 2. In this figure the asymptotic behavior of T_b is obvious if $1 > N > 0$. For a given value of N and for comparatively small values of b, an increment in b increases T_b. If b is increased beyond a certain value, the value of T_b approaches asymptotically to the value of d_1 expressed in the equation

$$d_1 = (N)^{-1/3}. \tag{52}$$

If T_b is smaller than d_1, β_2 is smaller than 1. If T_b equals d_1, β_2 is equal to 1 and a_3 given by equation (48) is not defined; consequently a_9–a_4 expressed in equations (42)–(47) are also undefined. Therefore, the temperature distribution in the fin cannot be predicted with equation (21). As stated previously, $T_b = 1$ when $N = 1$. It follows from equation (41) that the condition in which $T_b = d_1$ is identical to the condition that $N = 1$. The foregoing implies that the temperature in the fin is constant if $T_b = d_1$. Consequently T_b is always less than d_1 and β_2 less than 1. If $N = 0$, β_2 is equal to -1 and α to $\pi/4$, and μ_b given by equation (28c) reduces to $\arccos(-1/T_b)$ which implies no restriction on the magnitude of T_b (i.e. $1 \leqslant T_b < \infty$). It is concluded from the foregoing that $1 \geqslant \beta_2 \geqslant -1$ if $1 \geqslant N \geqslant 0$.

Some values of T are missing in Table 2. This was because the desk calculator used to carry out the calculations computed internally using each number as a 10-digit mantissa and a two-digit exponent of 10 (see equations (47) and (48); $\beta_1 \simeq \beta_2$). In order to

Table 2. $f\sqrt{Bi}$ and dimensionless temperature in the fin as a function of dimensionless variables for $n = 2$

	$N = 0$					$N = 0.25$					$N = 0.50$					$N = 0.75$				
		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$		$X = 0$	$X = 0.25$	$X = 0.50$	$X = 0.75$
b	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T	$f\sqrt{Bi}$	T_b	T	T	T
0.10	0.289	1.047	1.026	1.011	1.003	0.217	1.035	1.019	1.008	1.002	0.144	1.023	1.013	1.006	1.001	0.072	1.011	1.006	1.003	1.001
0.25	0.409	1.107	1.059	1.026	1.006	0.306	1.078	1.043	1.019	1.005	0.203	1.050	1.028	1.012	1.003	0.101	1.024	1.013	1.006	1.001
0.50	0.503	1.193	1.104	1.045	1.011	0.374	1.136	1.073	1.031	1.008	0.247	1.085	1.046	1.020	1.005	0.122	1.040	1.022	1.009	1.002
1	0.583	1.331	1.173	1.073	1.018	0.429	1.220	1.114	1.048	1.012	0.281	1.132	1.068	1.028	1.007	0.138	1.060	1.030	1.013	1.003
2	0.641	1.540	1.270	1.111	1.027	0.464	1.325	1.159	1.064	1.015	0.300	1.181	1.087	1.034	1.008	0.146	1.078	1.037	1.014	1.003
3	0.664	1.706	1.340	1.138	1.033	0.475	1.390	1.181	1.071	1.016	0.305	1.207	1.093	1.035	1.008	0.148	1.087	1.038	1.014	1.003
5	0.683	1.974	1.444	1.174	1.041	0.482	1.465	1.196	1.072	1.016	0.308	1.233	1.092	1.032	1.007	0.149	1.094	1.035	1.012	1.002
7.5	0.693	2.243	1.537	1.206	1.048	0.485	1.514	1.195	1.067	1.014	0.308	1.246	1.085	1.026	1.005	0.149	1.098	1.031	1.009	1.002
10	0.698	2.470	1.607	1.229	1.053	0.486	1.540	1.186	1.060	1.012	0.309	1.252	1.076	1.021	1.004	0.149	1.099	1.027	1.007	1.001
20	0.704	3.175	1.790	1.284	1.064	0.486	1.576	1.144	1.036	1.006	0.309	1.259	1.051	1.010	1.001	0.149	1.100	1.017	1.003	1.000
50	0.706	4.577	2.046	1.374	1.077	0.486	1.587	1.072	1.010	1.001	0.309	1.260	—	—	1.000	0.149	1.100	—	—	1.000
100	0.707	6.156	2.233	1.397	1.086	0.486	1.587	—	—	1.000	0.309	1.260	—	—	1.000	0.149	1.100	—	—	1.000

determine these values, a double precision computer is required.

Fin effectiveness

The fin effectiveness is a proper criterion for the evaluation of the performance of a heat transfer surface with and without a fin. It is defined by

$$f = \frac{-K\left(\dfrac{d\theta}{dx}\right)_{x=0}}{h_b\theta_b} = \frac{-K\dfrac{\theta_e}{L}\left(\dfrac{dT}{dX}\right)_{X=0}}{h_b\theta_b} \qquad (53)$$

where the numerators give the heat flux and the denominators the heat flux at the surface in the absence of the fin.

In order to calculate f for any value of n excluding $n = -2$, the temperature gradient given by equation (17) is introduced into equation (53). After rearrangement and considering that $T = T_b$ for $X = 0$, the latter yields

$$f\sqrt{Bi} = \left\{\frac{2(T_b^{(n+2)} + \{n+2\}\{1-T_b\}NT_b^{(n+1)} - 1)}{(n+2)T_b^{(n+2)}}\right\}^{0.5} \qquad (54)$$

in which

$$Bi = h_b W/K = a\theta_b^n W/K. \qquad (55)$$

The definition of the Biot number given in the literature is identical to that expressed in equation (55) for the straight fin but different from it for the cylindrical fin.

The fin effectiveness is calculated with equation (54) after T_b is determined either with an analytic method or with a numerical method.

As noted previously, T_b has only one value if the values of b, N and n are fixed. It thus follows from equation (54) that $f\sqrt{Bi}$ also has one value if the values of b, N and n are fixed. Therefore, taking N as a parameter (i.e. $N = 0$, 0.25, 0.50 and 0.75), $f\sqrt{Bi}$ is plotted against b in Fig. 3 for $n = -1$ and 0 and in Fig. 4 for $n = 1$ and 2. For $N = 1$, the fin effectiveness is zero. This is deduced either from equation (53) (i.e. $dT/dX = 0$ for $X = 0$) or from equation (54) (i.e. $T_b = 1$).

Figures 3 and 4 reveal the conditions that for given values of b, N and Bi, an increase in n decreases the fin effectiveness and that for given values of b, n and Bi a decrease in N increases fin effectiveness.

It follows from the definition of the fin effectiveness that for the cylindrical fin and the straight fin of unit length in the longitudinal direction, the rate of heat flow conducted to the fin from its base is equal to $A\theta_b h_b f$. The rate of heat flow from the fin to the fluid surrounding it is then given by

$$G = A(\theta_b h_b f + QL). \qquad (56)$$

Extrapolation of the results obtained

As can be deduced from the foregoing, T is a function of b, N, n and X. Accordingly, it is only a function

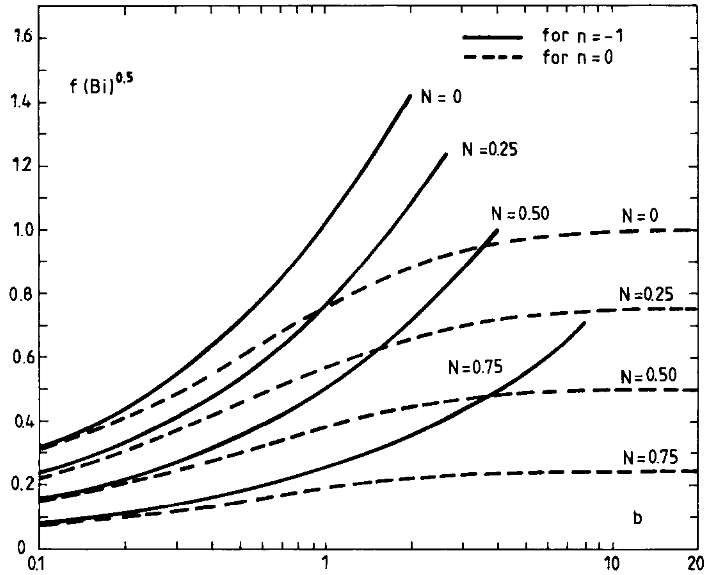

FIG. 3. $f\sqrt{Bi}$ as a function of b and N for $n = -1$ and 0.

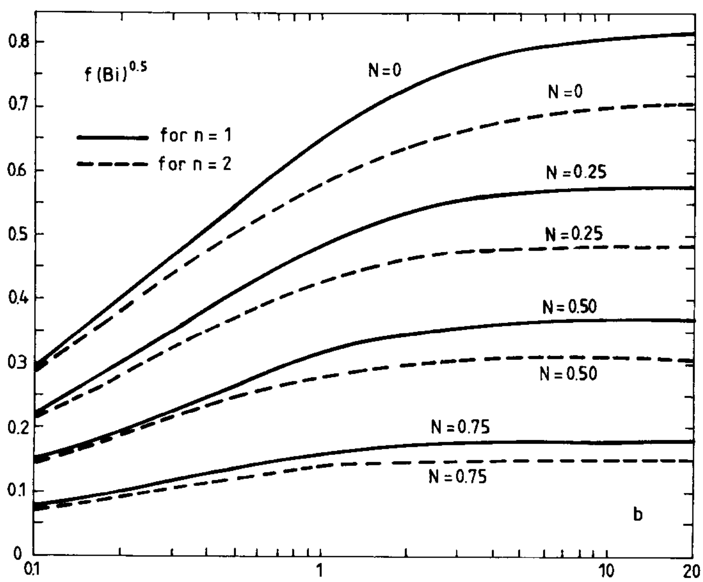

FIG. 4. $f\sqrt{Bi}$ as a function of b and N for $n = 1$ and 2.

of n if the values of b, N and X are fixed. Therefore, if n is a fraction between -1 and 2, the value of T for this n can be determined with interpolation using the values of T calculated for $n = -1$, 0, 1 and 2 and for given values of b, N and X. In Fig. 5, this interpolation method is illustrated for the prediction of T_b for $n = -0.5$, 0.4 and 1.6 whilst $b = 1.2$, $N = 0.15$ and $X = 0$.

Since $f\sqrt{Bi}$ is a function of b, N and n, a procedure similar to the foregoing interpolation method can be used for the determination of $f\sqrt{Bi}$ if n is a fraction between -1 and 2.

In order to further verify the method proposed, the following procedure was carried out. First for some randomly selected values of b, N and n, the values of T_b and of $f\sqrt{Bi}$ were calculated with the method. (For the design engineer, the evaluation of T_b and of $f\sqrt{Bi}$ are probably of the most practical significance.) Thereafter these values were compared with those predicted with a numerical method using equations (2) and (54). The results obtained are tabulated in Table 3 in which the columns designated I give the errors in predicting T_b and those designated II the errors in predicting $f\sqrt{Bi}$. The error was based on the value calculated with a numerical method, i.e. the fourth-order Runge–Kutta procedure. If both, the value of T_b and of $f\sqrt{Bi}$ are required, it is sufficient to determine T_b since $f\sqrt{Bi}$ can be evaluated with equation (54) if T_b is known. In this case, the error in predicting $f\sqrt{Bi}$ improves, as can be deduced from the columns designated III in Table 3. The errors given in this table seem acceptable for most practical

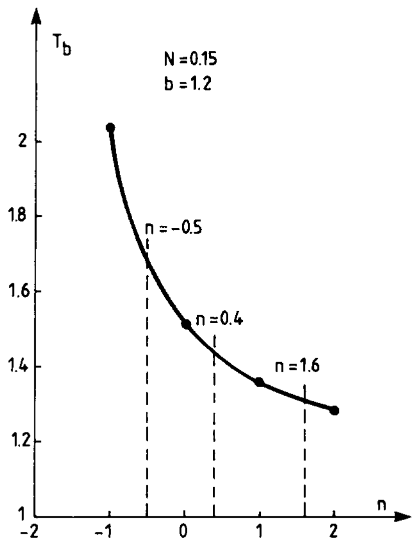

FIG. 5. Determination of T_b if n is a fraction between -1 and 2.

applications. If n is a fraction between -1 and 0 and b is greater than $2/(1-N)$ (see equation (13)), then the method proposed should fail to yield T_b or $f\sqrt{Bi}$.

SOLUTION OF THE DIFFERENTIAL EQUATION FOR NON-UNIFORM HEAT GENERATION

A cubic relationship for heat generation

Internal heat generation in the fin is assumed to be dependent on its temperature and this dependency is expressed by

$$Q = g_3 t_f^3 + g_2 t_f^2 + g_1 t_f + g \tag{57}$$

where g to g_3 are given dimensional constants. With proper algebraic manipulations, equation (57) can be reduced to

$$Q = S_3\theta^3 + S_2\theta^2 + S_1\theta + S \tag{58}$$

in which

$$S_3 = g_3 \tag{59}$$

$$S_2 = 3g_3 t_v + g_2 \tag{60}$$

$$S_1 = 3g_3 t_v^2 + 2g_2 t_v + g_1 \tag{61}$$

$$S = g_3 t_v^3 + g_2 t_v^2 + g_1 t_v + g. \tag{62}$$

In order to determine the temperature distribution in the fin with non-uniform internal heat generation, equation (2) will first have to be solved. (The boundary conditions given by equations (3) and (4) hold good.) To this end, Q expressed in equation (58) is introduced into N defined by equation (8). Using this N and after rearrangement, the first integration of equation (2), which satisfies the boundary condition given in equation (4), yields

$$\frac{\mathrm{d}T}{(qT^{(n+2)} - q_4T^4 - q_3T^3 - q_2T^2 - q_1T + 2C)} = -\mathrm{d}X \tag{63}$$

Table 3. Percentage error in the prediction of T_b and of $f\sqrt{Bi}$ with the interpolation method proposed

b	$N = 0$						$N = 0.15$						$N = 0.35$					
	$n = -0.5$			$n = 1.5$			$n = 0.4$			$n = 1.6$			$n = 0.6$			$n = 0.14$		
	I	II	III	I	II	III	I	II	III	I	II	III	I	II	III	I	II	III
1.2	1.9	2.0	0.9	−0.6	0.0	−0.4	−0.3	−0.5	−0.3	0.2	0.6	0.1	−0.1	−0.2	0.0	−0.3	−0.7	−0.4
5.4	—	—	—	−0.5	−0.4	−0.1	−1.3	0.4	−0.1	0.6	0.0	0.1	0.6	0.0	0.1	0.0	0.0	0.1
14	—	—	—	−0.4	−0.7	0.0	−3.2	0.0	−0.1	1.9	−0.2	0.1	−1.8	0.7	−0.1	−0.3	−0.4	0.0

where

$$q_4 = \frac{bWS_3}{2aT_b^2\theta_b^{(n-2)}} \tag{64}$$

$$q_3 = 2bWS_2/\{3aT_b\theta_b^{(n-1)}\} \tag{65}$$

$$q_2 = S_1bW/(a\theta_b^n) \tag{66}$$

$$q_1 = 2bWST_b/\{a\theta_b^{(n+1)}\} \tag{67}$$

$$q = 2b/\{T_b^n(n+2)\} \tag{68}$$

$$C = (q_4+q_3+q_2+q_1-q)/2. \tag{69}$$

and q–q_4 are dimensionless constants.

With the exceptions of the conditions that $g_3 = g_2 = 0$ and $n = -1$ and that $g_3 = g_2 = 0$ and $n = 0$, the solution of equation (63) is again given by equation (21). T_b is determined from equation (24). In order to utilize equations (21) and (24), $\mu(T)$, α and m in equation (21) should be calculated. These depend on the roots of the polynomial equation in the denominator of equation (63) and can be evaluated with the formulae presented herein if the numerical values of g, g_1, g_2, g_3, t_v, θ_b, K, L, a, n and W are known. One of the roots of this polynomial equation is always equal to 1.

If the polynomial equation is a cubic equation, then the conditions that it has three real roots and $1 > \beta_2 > \beta_3$ and that it has one real and two complex roots are of practical significance as discussed previously for the case that $n = 1$ and Q is a constant. First the cubic equation is reduced to

$$\gamma_2[T^3+e_2T^2+e_1T+e] = 0 \tag{70}$$

where γ_2 and e–e_2 are dimensionless constants.

If the cubic equation has three real roots and $1 > \beta_2 > \beta_3$, $\mu(T)$, m and α are given by equations (25a), (26a) and (27a), respectively. The value of γ in equation (26a) is equal to γ_2 in equation (70). Equation (30a) and μ_b and μ_e expressed in equations (28a) and (29a) hold good.

If the cubic equation has one real and two complex roots, $\mu(T)$, m and α are given by equations (25b), (26b) and (27b), respectively. The value of γ in equation (26b) is equal to γ_2 in equation (70). M and r in μ, m and α are determined by establishing an equation similar to equation (32). Equation (30b) and μ_b and μ_e expressed in equations (28b) and (29b) hold good.

If the foregoing polynomial equation is a fourth-order equation, it is first reduced to

$$\gamma_3[T^4+e_3T^3+e_4T^2+e_5T+e_6] = 0 \tag{71}$$

where γ_3, e_3–e_6 are non-dimensional constants. One of the roots of equation (71) is 1. If internal heat generation in the fin is equal to zero, one of the remaining roots becomes -1 and equation (71) reduces to equation (39) provided that $N = 0$ in the latter. Therefore, it is concluded that the case being dealt with is similar to the case in which $n = 2$ and Q is a constant. Accordingly $\mu(T)$, m and α in equation (21) are given by equations (25c), (26c) and (27c), respectively. β_1 is equal to 1 and β_2 is smaller than β_1. γ_1 in equation (26c) is equal to γ_3 in equation (71). a_1 and a_2 in a_4–a_9 are determined by establishing an equation similar to equation (49). Equation (30c) and μ_b and μ_e expressed in equations (28c) and (29c) hold good.

The determination of the temperature distribution in the fin is carried out in a manner similar to that explained previously herein for the case wherein $n = 1$ and Q is a constant. The fin effectiveness is predicted with equation (53).

A linear relationship for heat generation

If g_2 and g_3 in equation (57) are equal to zero and if $n = -1$ and 0, the differential equation of the temperature distribution in the fin (i.e. equation (2)) reduces to

$$\frac{d^2T}{dX^2}+S_4T = S_5 \tag{72}$$

in which

$$S_4 = b\theta_bWS_1/a \quad \text{for } n = -1 \tag{73}$$

$$S_5 = bT_b(1-WS/a) \quad \text{for } n = -1 \tag{74}$$

$$S_4 = -b(1-WS_1/a) \quad \text{for } n = 0 \tag{75}$$

$$S_5 = -bWST_b/(a\theta_b) \quad \text{for } n = 0. \tag{76}$$

The boundary conditions expressed in equations (3) and (4) hold good and Q is given by

$$Q = S_1\theta+S \tag{77}$$

where S is the part of the internal heat generation corresponding to the condition that $\theta = 0$. Accordingly S_1 and S in this equation are positive. In equation (75) WS_1/a is the ratio of the heat flux due to a part of the internal heat generation to the total heat flux at location x (i.e. $WS_1\theta/a_1\theta$) and it is always smaller than 1. Consequently S_4 given by equation (75) is always negative. Considering now the foregoing, the solution of equation (72) is carried out in accordance with ref. [8]. This solution, which satisfies the boundary conditions expressed in equations (3) and (4), is given by equation (78) when $n = -1$

$$T = \left(T_b - \frac{S_5}{S_4}\right)\{\cos(\sqrt{S_4}X) + \tan\sqrt{S_4}\sin(\sqrt{S_4}X)\} + \frac{S_5}{S_4} \tag{78}$$

and by equation (79) for $n = 0$

$$T = \left(T_b - \frac{S_5}{S_4}\right)\frac{\cosh\{\sqrt{-S_4}(1-X)\}}{\cosh\sqrt{-S_4}} + \frac{S_5}{S_4}. \tag{79}$$

T_b is determined using the condition that $T = T_e = 1$ for $X = 1$. It is expressed in equation (80) for $n = -1$

$$T_b = \{(1-S_6)(\cos\sqrt{S_4}+\tan\sqrt{S_4}\sin\sqrt{S_4})+S_6\}^{-1} \tag{80}$$

and in equation (81) for $n = 0$

$$T_b = \left(\frac{1-S_7}{\cosh\sqrt{-S_4}} + S_7\right)^{-1} \tag{81}$$

in which

$$S_6 = \frac{(1 - WS/a)a}{\theta_b WS_1} \tag{82}$$

$$S_7 = \frac{WS}{a\theta_b(1 - WS_1/a)}. \tag{83}$$

SUMMARY/CONCLUSIONS

Temperature distributions in a straight fin of rectangular profile (or in a cylindrical fin) with uniform and non-uniform internal heat-generation characteristics and non-uniform heat transfer coefficients are derived analytically. The heat transfer coefficient is assumed to be a power function of the difference between the temperature of the fin and that of the fluid surrounding it. The power of this function is taken as being equal to -1, 0, 1 and 2. Non-uniform internal fin heat generation is assumed to depend on the fin temperature and this dependency is expressed in a polynomial equation up to third degree.

The results obtained for uniform internal heat generation are reduced to tables and graphs for the convenience of the design engineer. For given values of the modified fin parameter, the generation number and the power in the heat transfer coefficient, these graphs yield the fin effectiveness for a given value of the modified Biot number and the dimensionless temperature at the fin base. An interpolation method is proposed to determine the dimensionless temperature in the fin and the fin effectiveness if the foregoing quoted power is a fraction between -1 and 2. For given values of the modified fin parameter, the generation number and the modified Biot number, an increase in the said power decreases the fin effectiveness. For given values of the modified fin parameter, the said power and the modified Biot number, a decrease in the generation number increases the fin effectiveness. The asymptotic behaviour of the dimensionless temperature at the fin base is illustrated.

The analytic solutions presented for the second-order non-linear differential equations may also be useful in other fields of engineering.

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DISTRIBUTIONS DE TEMPERATURE DANS DES AILETTES AVEC GENERATION DE CHALEUR UNIFORME OU NON ET COEFFICIENT DE TRANSFERT NON UNIFORME

Résumé—On détermine analytiquement la distribution de température dans une ailette droite de profil rectangulaire (ou dans une ailette circulaire) avec une génération de chaleur uniforme ou non et un coefficient de transfert thermique non uniforme. Le coefficient de transfert est supposé être une fonction puissance de la différence entre la température de l'ailette et celle du fluide environnant. La puissance est prise égale à -1, 1 et 2. La génération de chaleur interne est supposée dépendre de la température de l'ailette et cette dépendance est exprimée par un polynôme du troisième degré. Les résultats obtenus pour la génération uniforme de chaleur sont présentés sous forme graphique et tabulée et une méthode d'interpolation est proposée pour déterminer la température de l'ailette et l'efficacité de l'ailette si la puissance est fractionnaire entre -1 et 2.

TEMPERATURVERTEILUNG IN RIPPEN BEI EINHEITLICHER UND UNEINHEITLICHER WÄRMEENTWICKLUNG UND UNEINHEITLICHEM WÄRMEÜBERGANGSKOEFFIZIENTEN

Zusammenfassung—Es werden die Temperaturverteilungen in einer geraden Rippe mit rechteckförmigem Profil (oder in einer zylinderförmigen Rippe) bei einheitlicher und uneinheitlicher innerer Wärmeentwicklung und uneinheitlichem Wärmeübergangskoeffizienten analytisch hergeleitet. Es wird angenommen, daß der Wärmeübergangskoeffizient als Potenzfunktion der Differenz zwischen der Temperatur an der Rippenoberfläche und der Temperatur des umgebenden Fluids dargestellt werden kann. Der Exponent der Funktion nimmt dabei die Werte -1, 0, 1 und 2 an. Zur Beschreibung der uneinheitlichen Wärmeentwicklung wird eine Abhängigkeit von der Rippenoberflächentemperatur angenommen und diese Abhängigkeit durch ein Polynom 3. Grades ausgedrückt. Die für gleichförmige innere Wärmeentwicklung gewonnenen Ergebnisse werden in tabellarischer und grafischer Form vorgestellt. Es wird ein Interpolationsverfahren vorgeschlagen, mit dem die Rippentemperatur und der Rippenwirkungsgrad bestimmt werden können, sofern der vorab angenommene Exponent im Bereich zwischen -1 und 2 liegt.

РАСПРЕДЕЛЕНИЕ ТЕМПЕРАТУРЫ В РЕБРАХ С ОДНОРОДНЫМ И НЕОДНОРОДНЫМ ТЕПЛОВЫДЕЛЕНИЕМ И ПЕРЕМЕННЫМ КОЭФФИЦИЕНТОМ ТЕПЛОПЕРЕНОСА

Аннотация—Аналитически получены распределения температуры в прямом ребре прямоугольного профиля (или в цилиндрическом ребре) с однородным и неоднородным внутренним тепловыделением и переменными коэффициентами теплопереноса. Полагается, что коэффициент теплопереноса является степенной функцией разности температур ребра и окружающей среды с показателем степени равным -1, 0, 1 и 2. Считается, что неоднородное внутреннее тепловыделение зависит от температуры ребра, эта зависимость выражается полиномами не выше третьей степени. Полученные результаты для однородного тепловыделения представлены таблицей и в виде графиков; в случае, когда показатель степени лежит в диапазоне от -1 до 2, для определения температуры ребра и его эффективности предложен интерполяционный метод.